Superposition and Conservation of Energy

Suppose you prepare a particle in a harmonic oscillator potential in a superposition of two energy stationary states, for example the $n = 0$ and the $n = 10$ states: $\Psi(\mathrm{x}, \mathrm{t}) = \frac{1}{\sqrt{2}}\left(\psi_0 e^{-iE_0 t/\hbar} + \psi_{10} e^{-iE_{10} t/\hbar}\right)$. Now measure the energy of the particle, which means that you apply the Hamiltonian operator $\widehat{\mathcal{H}}$. You will find either $E_0 = \frac{\hbar\omega}{2}$ or $E_{10} = \frac{21\hbar\omega}{2}$ with equal likelihood. In other words for an ensemble of identically prepared systems, one will find $\langle\widehat{\mathcal{H}}\rangle = \frac{1}{2}(E_0 + E_{10}) = \frac{11}{2}\hbar\omega$. OK, but suppose you measure the particle once and find an energy of $\frac{21\hbar\omega}{2} > \frac{11}{2}\hbar\omega$. Where did the extra energy come from?

This illustrates the non-classical nature of a quantum superposition. First, note that the superposition state is NOT a determinate state for the Hamiltonian. It does not have a fixed value of energy. One is not justified in asking "where does the energy go/come-from?" upon measuring the energy of the particle prepared in a superposition of two or more energy eigenstates.

But what about energy conservation? It survives, but only in an ensemble average form. Consider the generalized Ehrenfest theorem:

$$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\widehat{\mathcal{H}}, \hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle \text{ with } \hat{Q} = \widehat{\mathcal{H}}\text{, then}$$

$$\frac{d}{dt}\langle\widehat{\mathcal{H}}\rangle = \frac{i}{\hbar}\langle[\widehat{\mathcal{H}}, \widehat{\mathcal{H}}]\rangle + \langle\frac{\partial\widehat{\mathcal{H}}}{\partial t}\rangle = 0 + 0 = 0$$

So "energy conservation" applies only to the Hamiltonian operator expectation value.

<u>Technical detail</u>: Note that $\Psi(\mathrm{x}, \mathrm{t}) = \frac{1}{\sqrt{2}}\left(\psi_0 e^{-iE_0 t/\hbar} + \psi_{10} e^{-iE_{10} t/\hbar}\right)$ is a solution to the time-dependent Schrodinger equation, but $\psi(\mathrm{x}) = \frac{1}{\sqrt{2}}(\psi_0 + \psi_{10})$ is <u>not</u> a solution to the time-independent Schrodinger equation! It does not satisfy the equation $\widehat{\mathcal{H}}\psi = E\psi$. Try it and see.